Exact resolution of the sparse spectral unmixing problem

Application of branch-and-bound algorithm

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The short λ's resume

2015 - 2018 Bachelor's degree in computer science and mathematics (Université de Nantes).

2018 - 2020 Master's degree in computer science (Université de Nantes & Université Libre de Bruxelles);
Major in Optimization in Operations Research;
Master's thesis : Branch-and-bound design for ℓ_0-*norm* optimization, application to the spectral unmixing problem;
Advisors : S.Bourguignon, R.Ben Mhenni

2020 - 2021 Project engineer (LS2N): algorithm implementation and scientific article redaction.

2021 - 2024 PhD student : Tomographic image reconstruction for a new uncommon 3-photons imaging system (CHU, Centrale Nantes, LS2N);
Advisors : Simon Stute, Jérôme Idier & Thomas Carlier

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Hyperspectral imaging

Let us consider a scene,

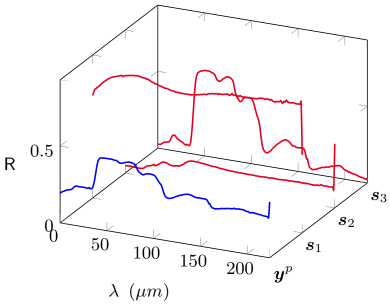
- Extracts information about the **present electromagnetic spectra** over a **large range of spectral bands**;
- A pixel $\Rightarrow$ the observed reflectance spectrum is a **mixture** formed by **different contributions** of the materials **spectral signatures**.

Spectral unmixing problem (SU)

- A **blind source separation** problem i.e. estimate both **spectral signatures** - aka. *atoms* or *endmembers* - and **their proportions** aka. *abundances*;
- **Supervised SU :** the mixture is searched in a known **dictionary** of reference spectra.

Assumption : Linear Mixing Model (LMM)

The **observed mixture** for a pixel can be expressed as the **linear combination of endmembers weighted by the fractional** abundances of each atom.



$\Rightarrow$ For a given pixel observed on N_λ spectral bands :

$$\boldsymbol{y} = \mathbf{S}\,\boldsymbol{a} + \boldsymbol{\varepsilon} \;\Leftrightarrow\; \boldsymbol{y} = \sum_{q \in [\![1,Q]\!]} a_q\,\boldsymbol{s}_q + \boldsymbol{\varepsilon}$$

where

- $\boldsymbol{y} \in \mathbb{R}^{N_\lambda}$: the reflectance spectrum;
- $\mathbf{S} \in \mathbb{R}^{N_\lambda \times Q}$: the dictionary of Q atoms;
- $\boldsymbol{a} \in \mathbb{R}^{Q}$: the abundance vector;
- $\boldsymbol{\varepsilon} \in \mathbb{R}^{N_\lambda}$: additional noise vector;
- $\boldsymbol{s}_q, a_q$: the q-th atom and its abundance.

Sum-to-one & nonnegativity constraints

$$\boldsymbol{y} = \sum_{q \in [\![1,Q]\!]} a_q \, \boldsymbol{s}_q + \boldsymbol{\varepsilon} \qquad \text{(LMM)}$$

Motivation :

- To express these abundances $\boldsymbol{a}$ as percentages;
- An active endmember $\Rightarrow$ a nonnegative explanatory contribution to the observed spectrum;
- The reflectance spectrum is only a part of the emitted light (absorption by materials) $\sim$ Normalization of the observed abundances.

Abundance nonnegativity constraint

$$\boldsymbol{a} \geq 0 \iff a_q \geq 0 \quad \forall q \in [\![1,Q]\!]$$

Abundance sum-to-one constraint

$$\mathbf{1}_Q^\intercal \, \boldsymbol{a} = 1 \iff \textstyle\sum_{q \in [\![1,Q]\!]} a_q = 1$$

Remark :

- This SU formulation is well identified in the literature as **Fully Constrained Least Squares** problems [Heinz et al., 2001] which naturally produce solutions where **some abundances are zero**.
- in **complex problems**, the estimated abundances are **spread over a substantial number** of components.

Sparsity constraint

Motivations

- **Physical constraint** : the observed scene is composed of a **limited number** of materials;
- The FCLS formulation may **fail in locating** the true endmembers;
- **Enforcing the number of nonzero abundances** may merely **ensure a correct mixture interpretation**.

The support of $\boldsymbol{a}$

$$\mathbf{supp}(\boldsymbol{a}) := \{q \in [\![1, Q]\!] \,|\, a_q \neq 0\}$$

The counting function : the ℓ_0-"norm"

$$\|\,\boldsymbol{a}\,\|_0 := \mathbf{Card}\,(\mathbf{supp}(\boldsymbol{a}))$$

K-sparse solution

$$\boldsymbol{a} \in \mathbb{R}^Q \quad \text{s.t.} \quad \|\,\boldsymbol{a}\,\|_0 \leq K, \;\; K \in \mathbb{Z}_+^*$$

Sparse spectral unmixing problem (ℓ_0-SU)

$$\min_{\boldsymbol{a}\in[0,1]^Q} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}\,\boldsymbol{a} \right\|_2^2 \qquad \text{s.t.} \quad \|\,\boldsymbol{a}\,\|_0 \leq K, \quad \mathbf{1}_Q^\intercal\,\boldsymbol{a} = 1 \qquad (\mathcal{P}_{2/0})$$

where $K \in \mathbb{Z}_+^*$ the sparsity coefficient **set *a priori***.

It's a complex problem

- ℓ_0-"norm" is not differentiable, not continuous $\Rightarrow$ $\mathcal{P}_{2/0}$ is **not convex**;
- ℓ_0-"norm" minimization problems are $\mathcal{NP}$-Hard [Natarajan, 1995];
- Can be **only considered** for problems with a **limited amount of atoms**.

But in practice

- The **number of active atoms** in a mixture is relatively **small** e.g. $K \in [\![1, 6]\!]$;
- The **dictionary** is typically composed of $10 \sim 500$ spectral signatures;
- The **sum-to-one & nonnegativity constraints restrict** the space of feasible solutions.

A short state of the art

Approximation: greedy algorithms using heuristics, convex or nonconvex relaxation [Tropp and Wright, 2010]

- **Reduce** the complexity of an **exhaustive exploration**;
- **Rapidly** provide sparse solution;
- The obtained results are **often suboptimal**.

Standard ℓ_0 problems using MIP techniques:
[Bourguignon et al., 2016], [Bertsimas et al., 2016];

Dedicated branch-and-bound algorithm to tackle ℓ_0 problems:
[Bienstock, 1999],[Bertsimas and Shioda, 2009], [Ben Mhenni, 2020], [Hazimeh et al., 2021];

Our approach

Implementing an **exact optimization algorithm** for the ℓ_0-SU

Exact optimisation of the ℓ_0-SU as a MIP [Ben Mhenni et al., 2018]

$$\min_{\boldsymbol{a}\in[0,1]^Q} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}\,\boldsymbol{a} \right\|_2^2 \qquad \text{s.t.} \quad \|\,\boldsymbol{a}\,\|_0 \le K, \quad \mathbf{1}_Q^\intercal\,\boldsymbol{a} = 1 \tag{$\mathcal{P}_{2/0}$}$$

MIP formulation

Introduce a binary decision variables: $\boldsymbol{b} \in \{0,1\}^Q$ s.t. $a_q = 0 \Leftrightarrow b_q = 0 \;\; \forall q \in [\![1, Q]\!]$;

Link $\boldsymbol{a}$ and $\boldsymbol{b}$: $|\,\boldsymbol{a}\,| \le M\,\boldsymbol{b}$ with $M \in \mathbb{R}_+^*$ aka. *bigM assumption*;

Trivially with $\boldsymbol{a} \in [0,1]$: $M = 1$ and $\boldsymbol{a} \le \boldsymbol{b}$

Rewrite the ℓ_0-"norm": $\|\boldsymbol{a}\|_0 = \sum_{q\in[\![1,Q]\!]} b_q$

$$\min_{\substack{\boldsymbol{a}\in[0,1]^Q \\ \boldsymbol{b}\in\{0,1\}^Q}} \quad \frac{1}{2}\left\|\boldsymbol{y} - \mathbf{S}\boldsymbol{a}\right\|_2^2 \qquad \text{s.t.} \quad \sum_{q\in[\![1,Q]\!]} b_q \le K, \quad \mathbf{1}_Q^\intercal\,\boldsymbol{a} = 1, \quad \boldsymbol{a} \le \boldsymbol{b} \tag{$\mathcal{P}_{2/0}$}$$

$\Rightarrow$ Just use IBM Cplex solver . . .

Exact optimisation of the ℓ_0-SU - Our approach

Our approach

- Implementing a **dedicated** and **exact optimization algorithm** for the ℓ_0-SU
  $\Rightarrow$ Branch-and-bound algorithm;
- Try to be ~~*harder*~~, *better*, *faster*, *stronger* than a commercial solver [Ben Mhenni, 2020];
- Provide an **open-source** software to the SU community.

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Branch-and-bound algorithm [Land and Doig, 1960]

OR's Mantra

When an exhaustive search of all feasible solutions is inconceivable, take a break and implement a branch-and-bound.

Main principle

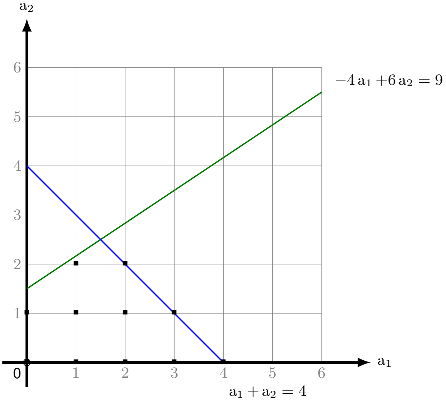
- Divide & conquer approach;
- Use of bounds on the optimal value to avoid the exploration of some regions of the search space; for a minimization problem :

$$\underline{z} \leq z^* \leq \overline{z}$$

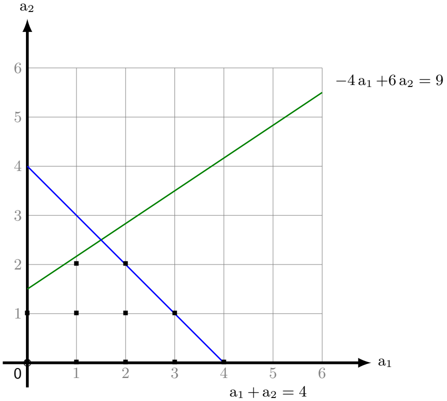
  The dual bound $\underline{z} \in \mathbb{R}$: a **lower bound** computed by a **relaxation**, often **easier to compute**;
  The primal bound $\overline{z} \in \mathbb{R}$: an **upper bound** given by any **feasible** solution.
- Build an **exploration tree** by repeating two steps: the **Branch** & the **Bound** operations.

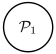
Let us consider:

$$(\mathcal{P}) : \min \left\{ f(\boldsymbol{a}) = \mathrm{a}_1 - 2\,\mathrm{a}_2 \;\middle|\; -4\,\mathrm{a}_1 + 6\,\mathrm{a}_2 \leq 9;\;\; \mathrm{a}_1 + \mathrm{a}_2 \leq 4;\;\; \boldsymbol{a} \in \mathbb{Z}_+^2 \right\}$$



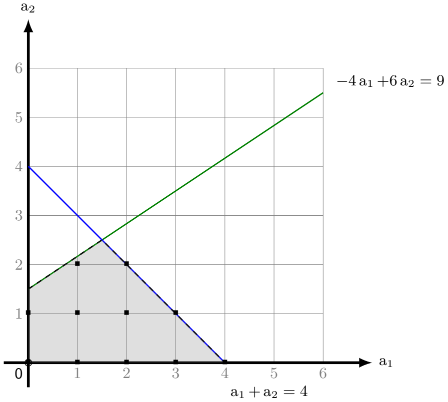
Our objective : find $\hat{\boldsymbol{a}} := \textbf{Argmin}(\mathcal{P})$

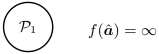




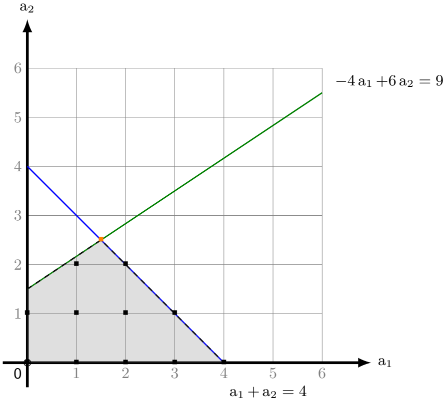
$f(\hat{\boldsymbol{a}}) = \infty$

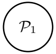
$$(\mathcal{P}_1) : \min \left\{ f(\boldsymbol{a}) = \mathrm{a}_1 - 2\,\mathrm{a}_2 \;\middle|\; -4\,\mathrm{a}_1 + 6\,\mathrm{a}_2 \leq 9;\;\; \mathrm{a}_1 + \mathrm{a}_2 \leq 4;\;\; \boldsymbol{a} \in \mathbb{R}_+^2 \right\}$$





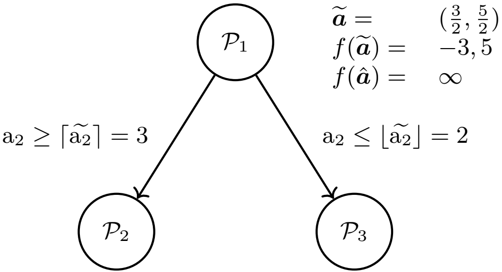
$$(\mathcal{P}_1) : \min \left\{ f(\boldsymbol{a}) = \mathrm{a}_1 - 2\,\mathrm{a}_2 \;\middle|\; -4\,\mathrm{a}_1 + 6\,\mathrm{a}_2 \leq 9;\;\; \mathrm{a}_1 + \mathrm{a}_2 \leq 4;\;\; \boldsymbol{a} \in \mathbb{R}_+^2 \right\}$$



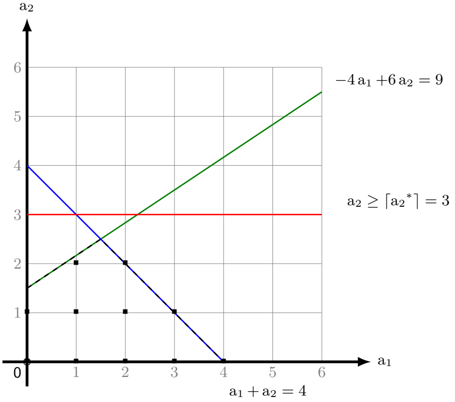


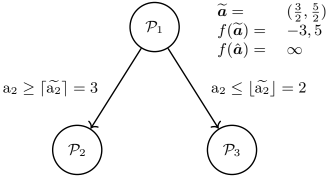
$\widetilde{\boldsymbol{a}} = \;(\frac{3}{2}, \frac{5}{2})$
$f(\widetilde{\boldsymbol{a}}) = \;-3,5$
$f(\hat{\boldsymbol{a}}) = \;\infty$

Branch : Use the optimum value to define inequalities and create a b-partition of the search space (exhaustive but not necessarily mutually exclusive);
e.g. Most infeasible branching $\Leftrightarrow$ most fractional component



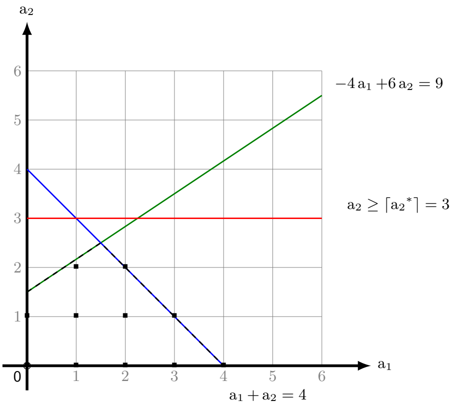
$$\mathcal{P}_2 := \mathcal{P}_1 \cap \{a_2 \geq 3\}$$

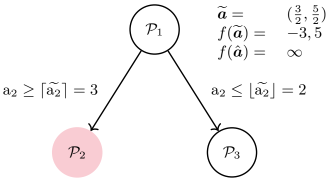




Branch-and-bound algorithm by the example

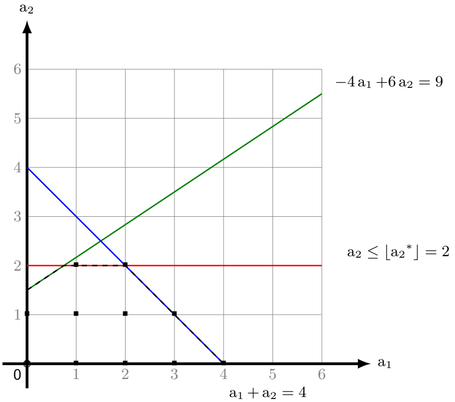
$$\mathcal{P}_2 := \mathcal{P}_1 \cap \{a_2 \geq 3\}$$

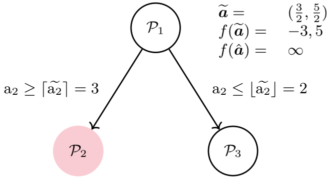




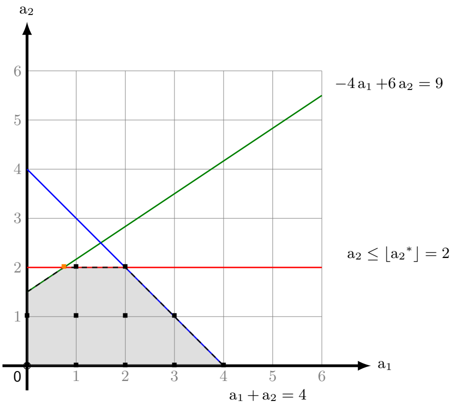
$\Rightarrow$ Pruning $\mathcal{P}_2$ by **nonfeasibility** !

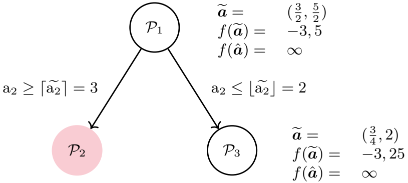
$$\mathcal{P}_3 := \mathcal{P}_1 \cap \{a_2 \leq 2\}$$





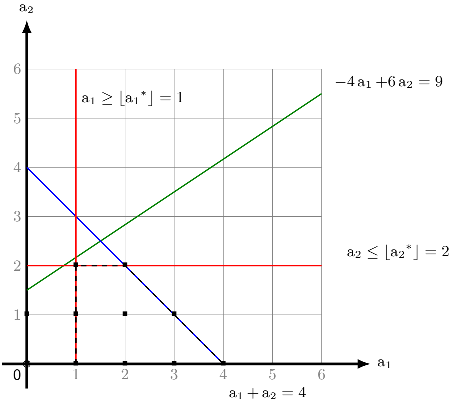
$$\mathcal{P}_3 := \mathcal{P}_1 \cap \{a_2 \le 2\}$$

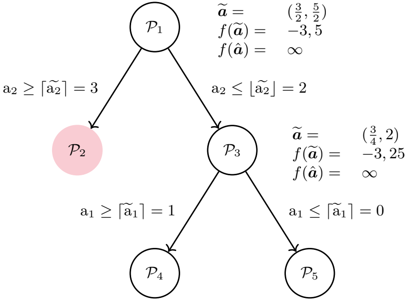




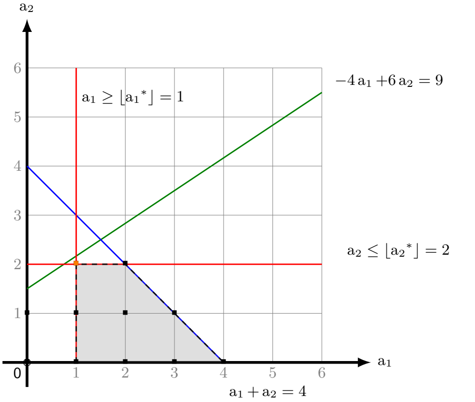
$\Rightarrow$ Nothing can be concluded: **Branch** on $\mathcal{P}_3$!

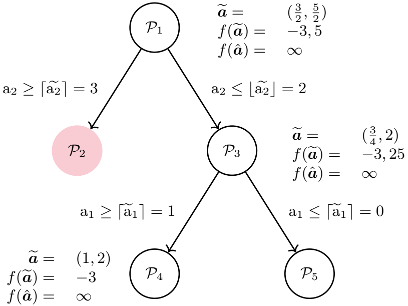
$$\mathcal{P}_4 := \mathcal{P}_1 \cap \{a_2 \leq 2\} \cap \{a_1 \geq 1\}$$





$$\mathcal{P}_4 := \mathcal{P}_1 \cap \{a_2 \leq 2\} \cap \{a_1 \geq 1\}$$

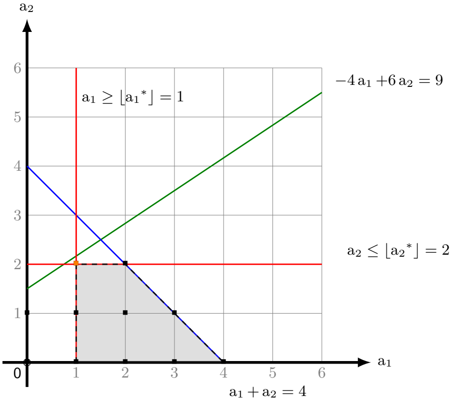


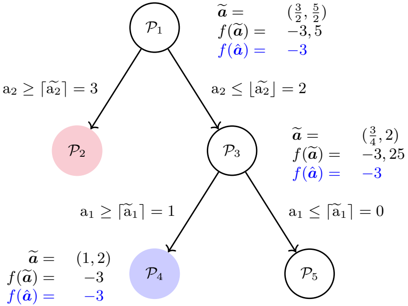


$\Rightarrow$ Prune $\mathcal{P}_4$ by **optimality** i.e. $\exists\, \widetilde{\boldsymbol{a}}$ $\mathbb{Z}_+$-feasible s.t. $f(\widetilde{\boldsymbol{a}}) < f(\hat{\boldsymbol{a}})$;
$\Rightarrow$ Update the global upper bound $f(\hat{\boldsymbol{a}})$

Branch-and-bound algorithm by the example

$$\mathcal{P}_4 := \mathcal{P}_1 \cap \{a_2 \leq 2\} \cap \{a_1 \geq 1\}$$



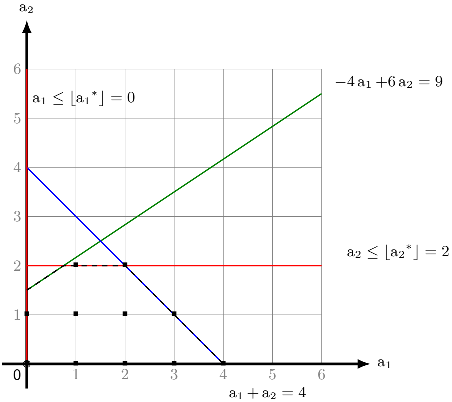


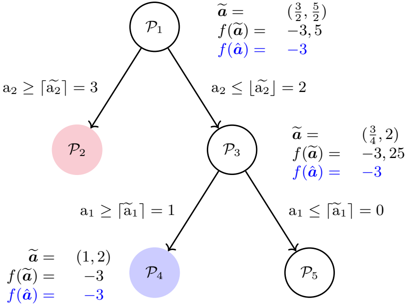
$\Rightarrow$ Prune $\mathcal{P}_4$ by **optimality** i.e. $\exists \widetilde{\boldsymbol{a}}$ $\mathbb{Z}_+$-feasible s.t. $f(\widetilde{\boldsymbol{a}}) < f(\hat{\boldsymbol{a}})$;

$\Rightarrow$ Update the global upper bound $f(\hat{\boldsymbol{a}})$

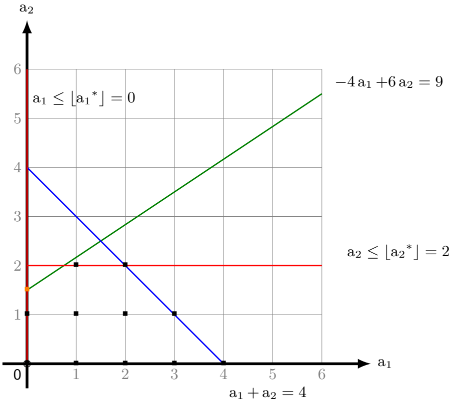
Branch-and-bound algorithm by the example

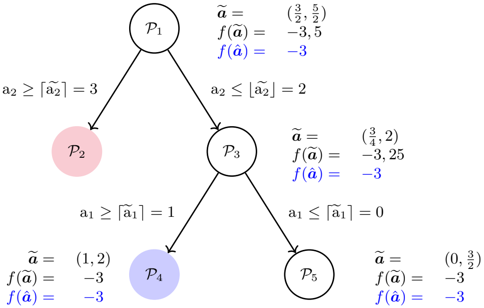
$$\mathcal{P}_5 := \mathcal{P}_1 \cap \{a_2 \leq 2\} \cap \{a_1 \leq 0\}$$



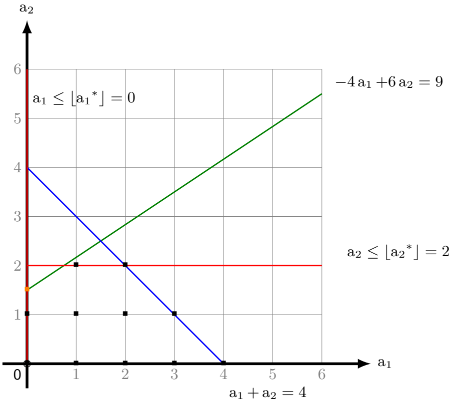


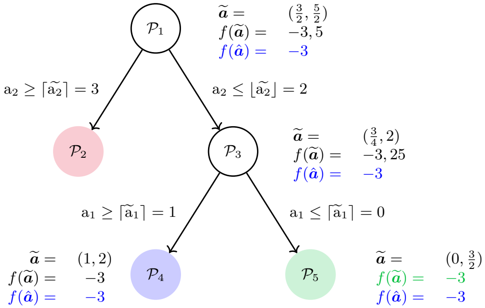
$$\mathcal{P}_5 := \mathcal{P}_1 \cap \{a_2 \le 2\} \cap \{a_1 \le 0\}$$





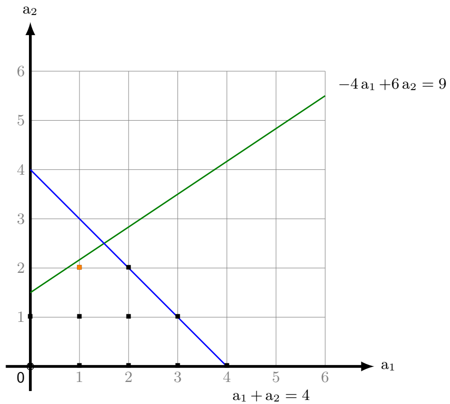
$$\mathcal{P}_5 := \mathcal{P}_1 \cap \{a_2 \le 2\} \cap \{a_1 \le 0\}$$

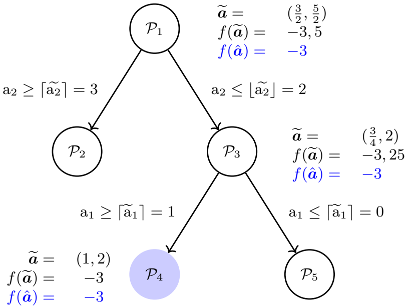




$\Rightarrow$ Prune $\mathcal{P}_5$ by **dominance** i.e. $\forall \widetilde{a}, f(\widetilde{a}) \ge f(\hat{a})$;

$$(\mathcal{P}) : \min \left\{ f(\boldsymbol{a}) = \mathrm{a}_1 - 2\,\mathrm{a}_2 \;\middle|\; -4\,\mathrm{a}_1 + 6\,\mathrm{a}_2 \leq 9;\;\; \mathrm{a}_1 + \mathrm{a}_2 \leq 4;\;\; \boldsymbol{a} \in \mathbb{Z}_+^2 \right\}$$





$\Rightarrow$ **Founded** and **certified** optimality : $\hat{\boldsymbol{a}} = (1, 2)$, $f(\hat{\boldsymbol{a}}) = -3$.

Branch-and-bound algorithm [Land and Doig, 1960]

Computational complexity: $\mathcal{O}(Tb^Q)$ with

b: *branching factor*

T: a fixed bound on time needed to explore the search (sub)spaces;

Q: the length of the longest path from the root to a leaf.

How to tune this algorithm ?

Branching strategy:

- Which value for the $b \in \mathbb{N}^*$?
- How to choose the branching variable a_i ?

Pruning rules:

- How to compute the lower bounds ?
- Add other valid inequality, cutting planes, dominance rules, ...

Search strategy:

- How to explore the tree ? e.g. DFS, BFS, ...

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$$\min_{\boldsymbol{a}\in[0,1]^Q} \quad \frac{1}{2}\left\|\, \boldsymbol{y} - \mathbf{S}\,\boldsymbol{a}\,\right\|_2^2 \quad \text{s.t.} \quad \|\,\boldsymbol{a}\,\|_0 \leq K, \quad \mathbf{1}_Q^\intercal\,\boldsymbol{a} = 1 \qquad (\mathcal{P}_{2/0})$$

The branch-and-bound will solve many subproblems on different subspaces :

Set notations

Let $\mathbb{S} := [\![1, Q]\!]$,

- The discarded variables: $\mathcal{Z} := \{q \in \mathbb{S} \,|\, a_q \notin \mathbf{supp}(\boldsymbol{a})\} \subset \mathbb{S}$;
- The remaining variables on $[0, 1]$: $\overline{\mathcal{Z}} := \mathbb{S} \setminus \mathcal{Z}$ with $\bar{n} := \mathbf{Card}(\overline{\mathcal{Z}})$
- The choosen variables: $\mathcal{C} \subset \overline{\mathcal{Z}}$

The **general formulation** of the subproblems :

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}}\in[0,1]^{\bar{n}}} \quad \frac{1}{2}\left\|\, \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}}\boldsymbol{a}_{\overline{\mathcal{Z}}}\right\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{a}_{\overline{\mathcal{Z}}}\|_0 \leq \widehat{K} - \mathbf{Card}(\mathcal{C}), \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$

ℓ_0-*norm* convex relaxation:

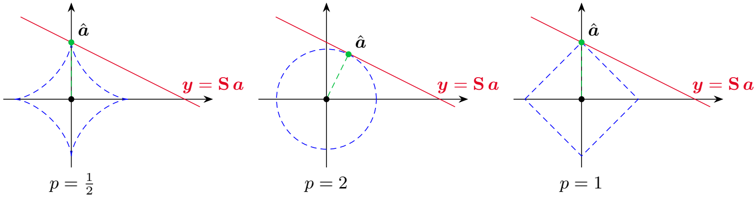


Figure: ℓ_p-norm $\forall p \in \{\frac{1}{2}, 2, 1\}$

Which one to choose?

Constraint : preserve the sparsity of the solutions;

- ℓ_p-*norm* with $p \in]0,1[$: sparse solution **but** nonconvex norm **and** we need to **ensure the optimality** of the solution
- ℓ_2-norm: nonsparse solution **but** convex norm;
- ℓ_1-norm: sparse solution **and** convex norm;

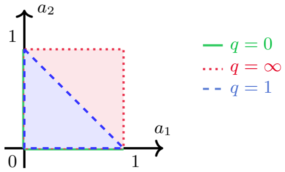
ℓ_0-*norm* convex relaxation:

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{a}_{\overline{\mathcal{Z}}}\|_0 \leq K - \mathbf{Card}(\mathcal{C}), \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\intercal} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$

Rewrite the sparsity constraint:

Using $\mathcal{B}_p^K := \left\{ \boldsymbol{a} \in \mathbb{R}^Q \mid \| \boldsymbol{a} \|_p \leq K \right\}$, $K \in \mathbb{Z}_+^*$: $\quad \boldsymbol{a} \in \left([0,1]^{\overline{n}} \cap \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})} \right)$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\intercal} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$



Property : the ℓ_1 relaxation

Given $K \in \mathbb{N}^*$ and under bound constraints:

$$\mathbf{conv}\left([0,1]^Q \cap \mathcal{B}_0^K\right) = \left([0,1]^Q \cap \mathcal{B}_1^K\right)$$

ℓ_0-*norm* convex relaxation:

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\mathsf{T}} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_1^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\mathsf{T}} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/1})$$

Beware of sparsity constraint !!

When $K > \mathbf{Card}(\mathcal{C})$, the relaxed sparsity constraint is **dominated** by the sum-to-one constraint; it can be **withdrawn** of $\mathcal{P}_{2/1}$.

$\rightsquigarrow$ $\mathbf{1}_{\overline{\mathcal{Z}}}^{\mathsf{T}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \leq K - \mathbf{Card}(\mathcal{C}) \ \wedge \ K > \mathbf{Card}(\mathcal{C}) \implies K - \mathbf{Card}(\mathcal{C}) > 1 \ \wedge \ \mathbf{1}_{\overline{\mathcal{Z}}}^{\mathsf{T}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \leq 1$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \cancel{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_1^{K - \mathbf{Card}(\mathcal{C})}}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\mathsf{T}} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/1})$$

$\Rightarrow$ Just another FCLS formulation.

ℓ_0-"norm" subproblem:

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\intercal} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \tag{$\mathcal{P}_{2/0}$}$$

ℓ_1-"norm" subproblem:

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}}^{\intercal} \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \tag{$\mathcal{P}_{2/1}$}$$

Main idea

To select an atom to add in the support of $\mathcal{P}_{2/0}$:

- Compute $\hat{\boldsymbol{a}} := \mathbf{Argmin}(\mathcal{P}_{2/1})$
- Choose the q-th endmember

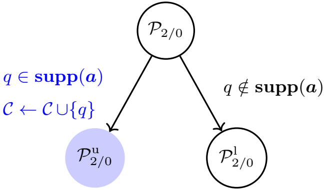
$$q := \mathbf{argmax}_{i \in \overline{\mathcal{Z}} \setminus \mathcal{C}} (\hat{a}_i)$$

- Apply a binary branching strategy on the q-th endmember, i.e. $\forall\, q \in \overline{\mathcal{Z}} \setminus \mathcal{C}$

$$q \in \mathbf{supp}(\boldsymbol{a}) \quad \textbf{or} \quad q \notin \mathbf{supp}(\boldsymbol{a})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \hat{\boldsymbol{a}}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/1})$$



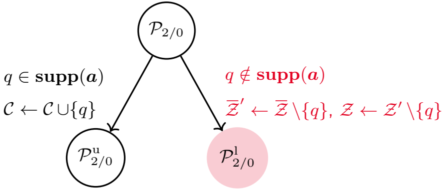
$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - (\mathbf{Card}(\mathcal{C})+1)}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0}^{\mathrm{u}})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\overline{n}}} \quad \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \hat{\boldsymbol{a}}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/1}^{\mathrm{u}})$$

$\Longrightarrow$ Same relaxations, **Nothing to do**.

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\bar{n}}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \boldsymbol{a}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/0})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}} \in [0,1]^{\bar{n}}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}} \, \boldsymbol{a}_{\overline{\mathcal{Z}}} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}}^\intercal \, \hat{\boldsymbol{a}}_{\overline{\mathcal{Z}}} = 1 \qquad (\mathcal{P}_{2/1})$$

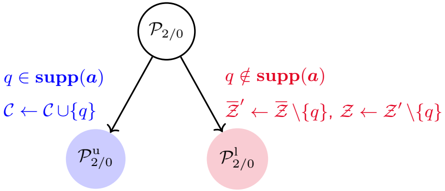


$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}'} \in [0,1]^{\bar{n}'}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}'} \boldsymbol{a}_{\overline{\mathcal{Z}}'} \right\|_2^2 \quad \text{s.t.} \quad \boldsymbol{a}_{\overline{\mathcal{Z}}'} \in \mathcal{B}_0^{K - \mathbf{Card}(\mathcal{C})}, \quad \mathbf{1}_{\overline{\mathcal{Z}}'}^\intercal \boldsymbol{a}_{\overline{\mathcal{Z}}'} = 1 \qquad (\mathcal{P}_{2/0}^{\mathrm{l}})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}'} \in [0,1]^{\bar{n}'}} \frac{1}{2}\left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}'} \, \boldsymbol{a}_{\overline{\mathcal{Z}}'} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}'}^\intercal \, \boldsymbol{a}_{\overline{\mathcal{Z}}'} = 1 \qquad (\mathcal{P}_{2/1}^{\mathrm{l}})$$

$\Longrightarrow$ Different relaxations, **A new FCLS problem**.

Additional assumption: $\hat{\boldsymbol{a}} := \mathbf{Argmin}(\mathcal{P}_{2/1})$ and $\| \hat{\boldsymbol{a}} \|_0 = K - 1$



$$\min_{\boldsymbol{a}_{\mathcal{C}} \in [0,1]^K} \frac{1}{2} \left\| \boldsymbol{y} - \mathbf{S}_{\mathcal{C}} \boldsymbol{a}_{\mathcal{C}} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\mathcal{C}}^{\intercal} \boldsymbol{a}_{\mathcal{C}} = 1 \qquad (\mathcal{P}^{\mathrm{u}}_{2/1})$$

$$\min_{\boldsymbol{a}_{\overline{\mathcal{Z}}'} \in [0,1]^{\overline{n}'}} \frac{1}{2} \left\| \boldsymbol{y} - \mathbf{S}_{\overline{\mathcal{Z}}'} \boldsymbol{a}_{\overline{\mathcal{Z}}'} \right\|_2^2 \quad \text{s.t.} \quad \mathbf{1}_{\overline{\mathcal{Z}}'}^{\intercal} \boldsymbol{a}_{\overline{\mathcal{Z}}'} = 1 \qquad (\mathcal{P}^{\mathrm{l}}_{2/1})$$

- $\mathcal{P}^{\mathrm{l}}_{2/1}$: to gather the best subset of atoms in the search subspace and get **feasible nonsparse solutions** i.e. **lower** bounds;
- $\mathcal{P}^{\mathrm{u}}_{2/1}$: to fit the optimal abundances of the selected $\mathbf{supp}(\boldsymbol{a})$ and get K**-sparse feasible solutions** i.e. **upper** bounds.

The dedicated branch-and-bound

Lower bound evaluations:

- Right child:
  $\Rightarrow$ Solve $\mathcal{P}^{l}_{2/1}$ to get lower bound,
  $\Rightarrow$ try to fathomed the node;
- Left child $\wedge$ $K = \mathbf{Card}(\mathcal{C})$:
  $\Rightarrow$ Solve $\mathcal{P}^{u}_{2/1}$ to get upper bound,
  $\Rightarrow$ fathomed the node;
- Left child $\wedge$ $K > \mathbf{Card}(\mathcal{C})$:
  $\Rightarrow$ Just branch on $q \in \overline{\mathcal{Z}} \setminus \mathcal{C}$.

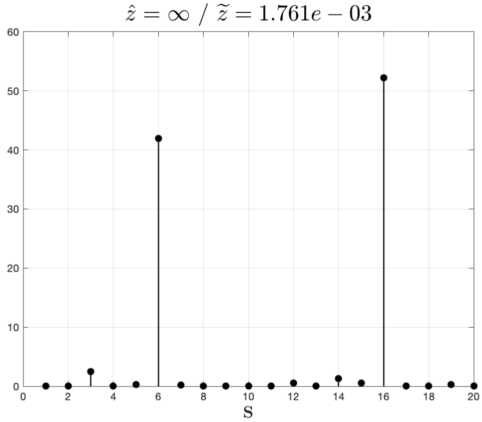
Search strategy:

DFS: to quickly find $\widehat{K}$-sparse solutions

Algorithme 1 : Dedicated branch-and-bound algorithm

Data : $\widehat{K}$, $\mathbf{S}$, $\boldsymbol{y}$,$\mathbb{S}$
Result : $(\hat{\boldsymbol{a}}, \hat{z})$
Initialization :
$\overline{\mathcal{Z}} \leftarrow \mathbb{S}$, $\mathcal{Z} \leftarrow \emptyset$, $\mathcal{C} \leftarrow \emptyset$, $n \leftarrow 0$, $L \leftarrow \emptyset$
$\hat{\boldsymbol{a}} \leftarrow \mathbf{0}_Q$, $\hat{z} \leftarrow \| \boldsymbol{y} \|_2^2$
push $\left(L, (\mathcal{C}, \overline{\mathcal{Z}}, \mathcal{Z}, n)\right)$
Main loop :
while $(L \neq \emptyset)$ **do**
  $(\mathcal{C}, \overline{\mathcal{Z}}, \mathcal{Z}, n) \leftarrow \mathbf{pop}(L)$
  $\text{terminal} \leftarrow \text{false}$
  Node evaluation procedure :
  $(\tilde{\boldsymbol{a}}, \tilde{z}) \leftarrow \mathbf{solve}(\mathcal{P}^f_{\text{n}}, \mathcal{C}, \overline{\mathcal{Z}}, \mathcal{Z})\ \forall f \in \{u, l, b\}$
  Bound procedure :
  if $(\tilde{z} \geq \hat{z})$ **then** // Fathomed by dominance
    $\text{terminal} \leftarrow \text{true}$
  else if $(\tilde{z} < \hat{z})$ **then** // Fathomed by optimality
    $\text{terminal} \leftarrow \text{true}$, $\hat{\boldsymbol{a}} \leftarrow \tilde{\boldsymbol{a}}$, $\hat{z} \leftarrow \tilde{z}$
  end
  Branch procedure :
  if $(\neg\, \textit{terminal} \wedge \textbf{\textit{Card}}(\bar{\mathbb{S}}) \neq \emptyset)$ **then**
    $q \leftarrow \mathbf{argmax}_{i \in \bar{\mathbb{S}}}(\hat{a}_i \,|\, \hat{a}_i > 0)$
    // Insert Right and Left children in L
    push $\left(L, \left(\mathcal{C}, \mathcal{Z} \cup \{q\}, \overline{\mathcal{Z}} \setminus \{q\}, 2n+2\right)\right)$
    push $\left(L, \left(\mathcal{C} \cup \{q\}, \mathcal{Z}, \overline{\mathcal{Z}} \setminus \{q\}, 2n+1\right)\right)$
  end
end

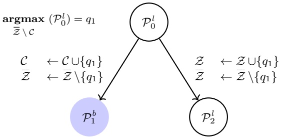
Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$

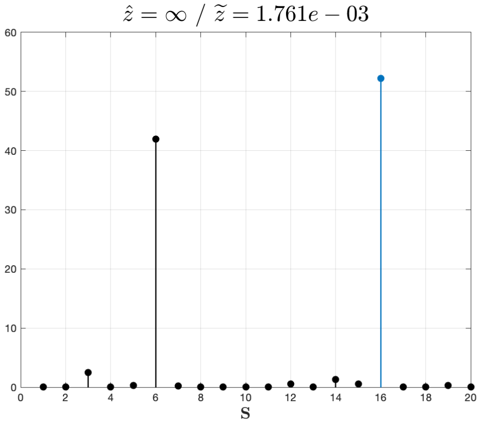


$\mathcal{C} = \emptyset$

The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$

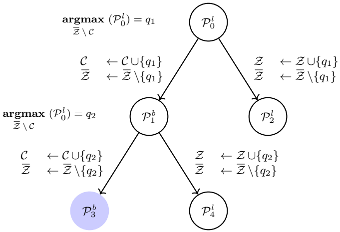


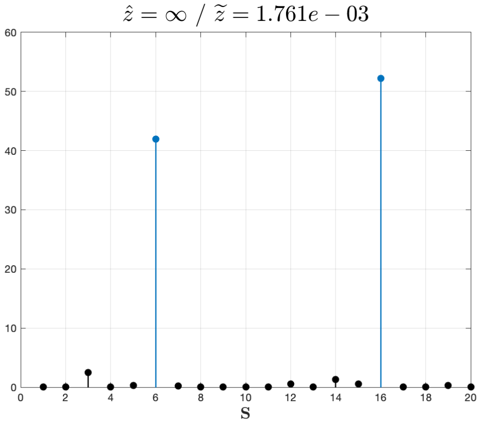


$\mathcal{C} = \{16\}$

The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$

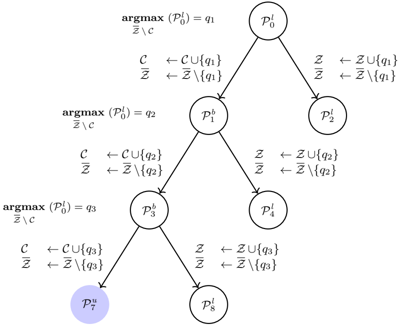


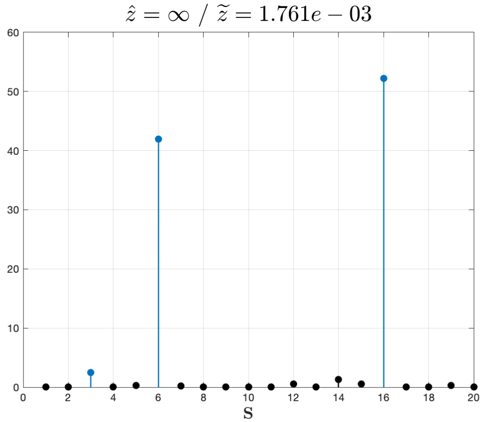


$\mathcal{C} = \{16, 6\}$

The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

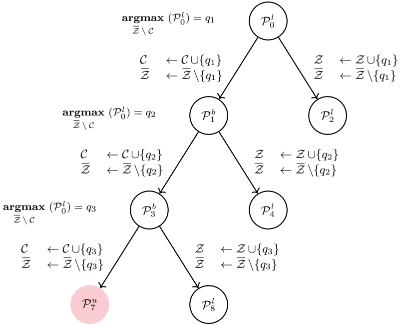
Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$





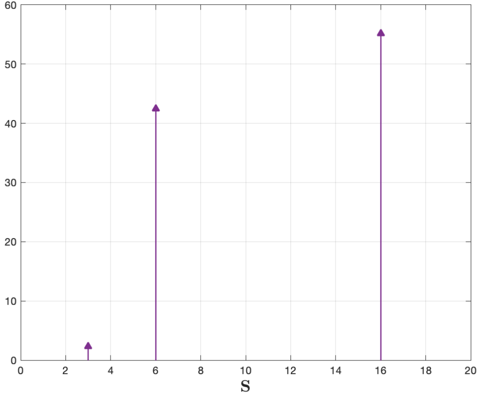
$$\mathcal{C} = \{16, 6, 3\} \quad \wedge \quad \textbf{Card}(\mathcal{C}) = K$$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



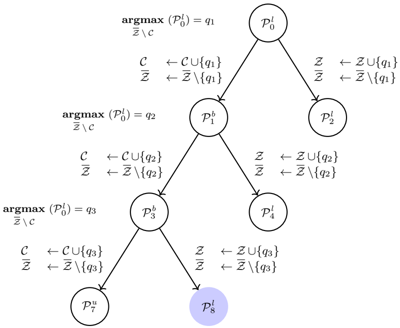
$\mathcal{C} = \{16, 6, 3\}$

$\hat{z} = \widetilde{z} = 1.789e - 03$

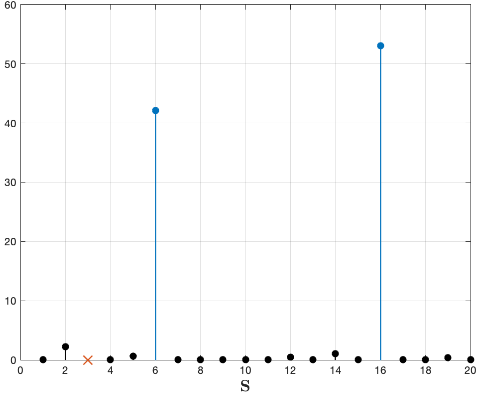


The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



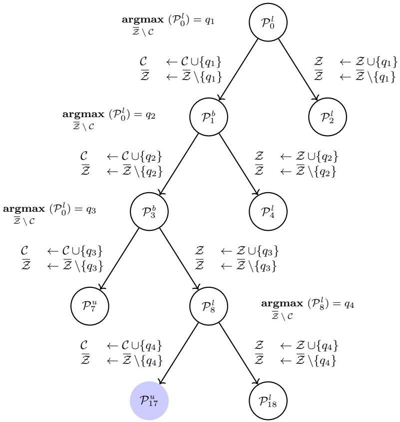
$\hat{z} = 1.789e-03 \ / \ \widetilde{z} = 1.781e-03 \ / \ \delta = 7.948e-06$

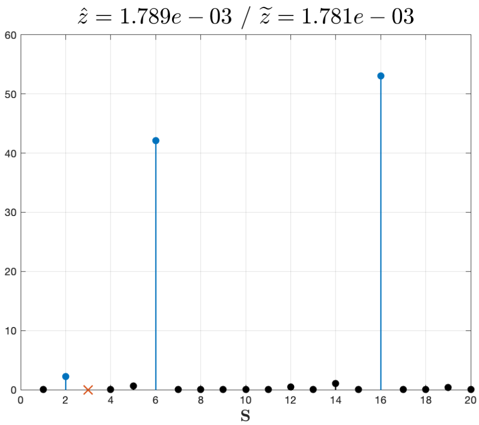


$\mathcal{C} = \{16, 6\}$

The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$

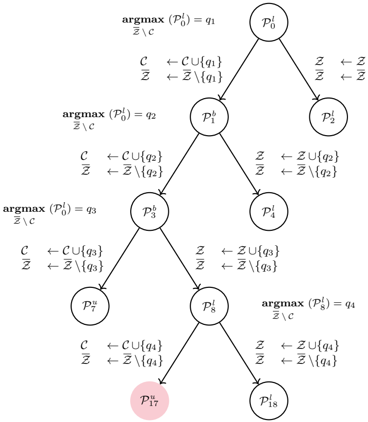




$$\mathcal{C} = \{16, 6, 2\} \quad \wedge \quad \textbf{Card}(\mathcal{C}) = K$$

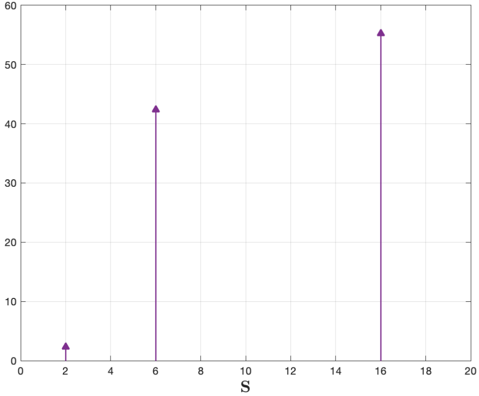
The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



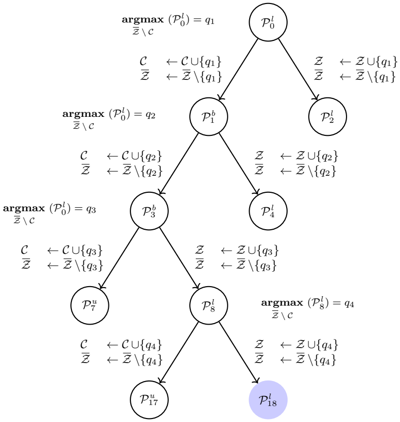
$\mathcal{C} = \{16, 6, 2\}$

$\hat{z} = 1.789e-03$ / $\widetilde{z} = 1.814e-03$ / $\delta = -2.500e-05$

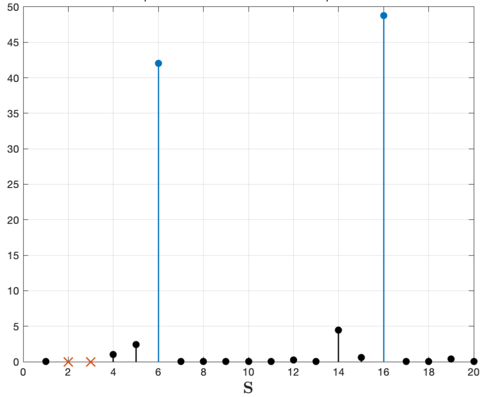


The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



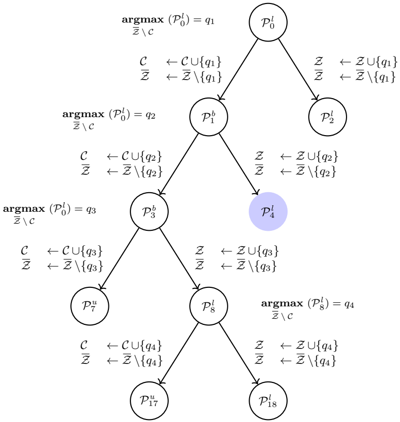
$\hat{z} = 1.789e - 03$ / $\widetilde{z} = 1.841e - 03$ / $\delta = -5.203e - 05$



$\mathcal{C} = \{16, 6\}$

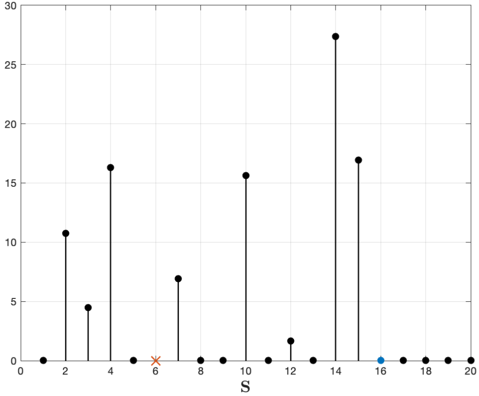
The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



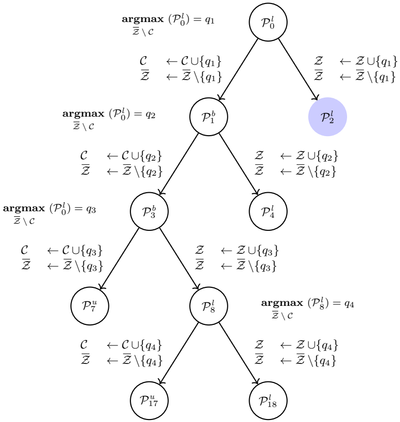
$\mathcal{C} = \{16\}$

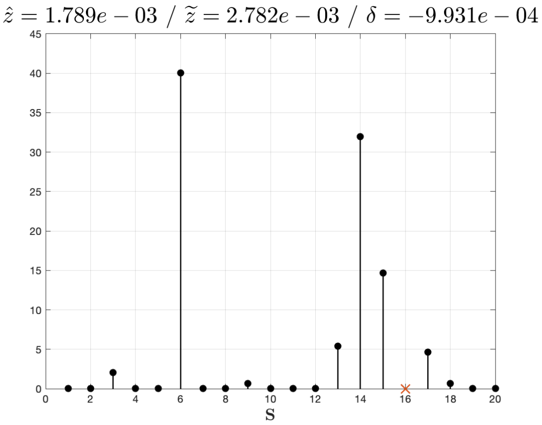
$\hat{z} = 1.789e - 03$ / $\widetilde{z} = 1.984e - 02$ / $\delta = -1.805e - 02$



The dedicated branch-and-bound: Example with $K = 3$, $Q = 20$

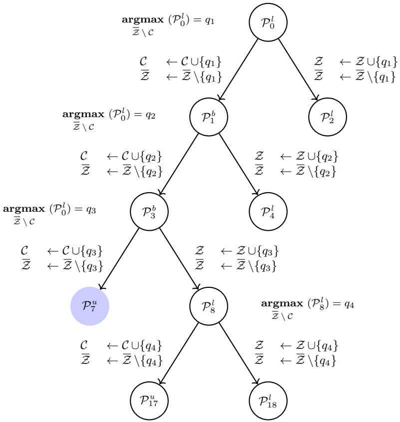
Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$





$\mathcal{C} = \emptyset$

Notations : $\hat{z}$ the best feasible solution, $\widetilde{z}$ the current solution, $\delta = \hat{z} - \widetilde{z}$



$\mathcal{C} = \{16, 6, 3\}$

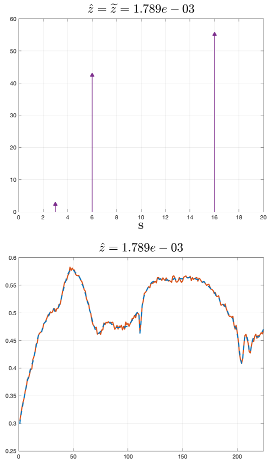


Table of contents

Spectral dictionary : USGS library [Clark et al., 2003]

Figure: Source : South Park - Season 18 (2014)

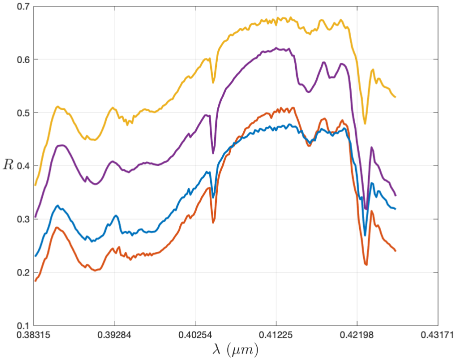


Figure: USGS atoms Calcite : AMX7, AMX18, AMX6, AMX43

USGS Digital Spectral Libraries

- $Q = 498$ pure spectral signatures (Minerals, Vegetation, Manmade materials, . . .)
- $N_\lambda = 224$ samples and the wavelength coverage is $[0.3, 2.7]$ μm

Simulated problems on USGS spectral library

Instances generation protocol

Two sets of designed problems such as:

Number ot atoms: $Q \in \{20, 50, 100, 200, 300, 400, 498\}$;

Sparsity level: $\widehat{K} \in [\![2, 6]\!]$;

Some noise: $\mathrm{SNR} \in \{40, 45, 50, 55, 60, \infty\}$ dB;

A minimum threshold: on the abundances $\tau = 0.1$;

Permutations of $\mathbf{S}$: 20.

Investigated methods

Time comparison:

B&B: our branch-and-bound with qpOASES solver [Ferreau et al., 2014] (C++)

MIP: using IBM Cplex solver (C++)

Quality comparison:

SDA: An iterative deflation algorithm [Greer, 2011] (Matlab)

$\widehat{K}$-FCLS: Two-phase algorithm using IBM Cplex solver (Matlab)

Simulated problems on USGS - Time comparison

$\mathrm{SNR} = \infty$		***B&B***	**MIP**	**Ratio**
$Q = 20$	$\widehat{K} = 2$	7,59e-04	8,23e-03	0,092
	$\widehat{K} = 4$	1,45e-03	6,17e-03	0,234
	$\widehat{K} = 6$	2,38e-03	6,70e-03	0,356
$Q = 100$	$\widehat{K} = 2$	1,70e-02	7,63e-02	0,223
	$\widehat{K} = 4$	1,86e-02	7,16e-02	0,26
	$\widehat{K} = 6$	3,64e-02	7,33e-02	0,497
$Q = 498$	$\widehat{K} = 2$	4,17e-01	6,95	0,06
	$\widehat{K} = 4$	1,03	9,28	0,11
	$\widehat{K} = 6$	1,93	1,91e+01	0,1

$\mathrm{SNR} = 60\mathrm{dB}$		***B&B***	**MIP**	**Ratio**
$Q = 20$	$\widehat{K} = 2$	6,46e-04	7,36e-03	0,088
	$\widehat{K} = 4$	1,45e-03	8,19e-03	0,177
	$\widehat{K} = 6$	2,17e-03	8,17e-03	0,265
$Q = 100$	$\widehat{K} = 2$	5,39e-03	7,61e-02	0,071
	$\widehat{K} = 4$	1,47e-02	7,68e-02	0,192
	$\widehat{K} = 6$	2,57e-02	1,63e-01	0,158
$Q = 498$	$\widehat{K} = 2$	2,79e-01	2,14	0,131
	$\widehat{K} = 4$	2,77	2,99	0,926
	$\widehat{K} = 6$	8,97	4,75	1,889

$\mathrm{SNR} = 50\mathrm{dB}$		***B&B***	**MIP**	**Ratio**
$Q = 20$	$\widehat{K} = 2$	5,76e-04	8,97e-03	0,064
	$\widehat{K} = 4$	1,15e-03	3,21e-02	0,036
	$\widehat{K} = 6$	1,64e-03	9,03e-03	0,182
$Q = 100$	$\widehat{K} = 2$	4,55e-03	9,41e-02	0,048
	$\widehat{K} = 4$	1,39e-02	7,67e-02	0,181
	$\widehat{K} = 6$	7,49e-02	1,84e-01	0,406
$Q = 498$	$\widehat{K} = 2$	5,29e-01	2,88	0,184
	$\widehat{K} = 4$	3,11e+01	1,74e+01	1,789
	$\widehat{K} = 6$	3,12e+02	1,51e+02	2,062

$\mathrm{SNR} = 40\mathrm{dB}$		***B&B***	**MIP**	**Ratio**
$Q = 20$	$\widehat{K} = 2$	6,91e-04	1,40e-02	0,049
	$\widehat{K} = 4$	1,37e-03	1,01e-02	0,136
	$\widehat{K} = 6$	2,18e-03	9,71e-03	0,225
$Q = 100$	$\widehat{K} = 2$	7,61e-03	1,10e-01	0,069
	$\widehat{K} = 4$	5,36e-02	1,35e-01	0,398
	$\widehat{K} = 6$	7,35e-01	4,18e-01	1,760
$Q = 498$	$\widehat{K} = 2$	7,95	4,91	1,619
	$\widehat{K} = 4$	4,39e+02	2,39e+02	1,834
	$\widehat{K} = 6$	9,08e+02	7,52e+02	1,208

- The exact method can be used to solve ℓ_0-SU problems
- The ratio of execution times seems to be favorable to our dedicated approach, except for the most difficult problems

Simulated problems on USGS - Quality comparison

Metric Exact Recovery Ratio (as %) : $\mathrm{ERR} := (\mathbf{supp}(\hat{\boldsymbol{a}}) \cap \mathbf{supp}(\mathring{\boldsymbol{a}}))\ /\ \widehat{K}$

$\mathrm{SNR} = \infty$		B&B	SDA	$\widehat{K}$-FCLS
$Q = 20$	$\widehat{K} = 2$	100	100	100
	$\widehat{K} = 4$	100	100	100
	$\widehat{K} = 6$	100	100	100
$Q = 100$	$\widehat{K} = 2$	100	100	100
	$\widehat{K} = 4$	100	100	100
	$\widehat{K} = 6$	100	100	100
$Q = 498$	$\widehat{K} = 2$	100	100	100
	$\widehat{K} = 4$	100	100	100
	$\widehat{K} = 6$	100	100	100

$\mathrm{SNR} = 60\mathrm{dB}$		B&B	SDA	$\widehat{K}$-FCLS
$Q = 20$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	100,0	100,0	100,0
$Q = 100$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	100,0	100,0	99,2
$Q = 498$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	100,0	99,6	98,3

$\mathrm{SNR} = 50\mathrm{dB}$		B&B	SDA	$\widehat{K}$-FCLS
$Q = 20$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	100,0	100,0	100,0
$Q = 100$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	100,0	99,6	97,9
$Q = 498$	$\widehat{K} = 2$	100,0	98,8	97,5
	$\widehat{K} = 4$	100,0	98,1	91,9
	$\widehat{K} = 6$	97,1	91,7	89,6

$\mathrm{SNR} = 40\mathrm{dB}$		B&B	SDA	$\widehat{K}$-FCLS
$Q = 20$	$\widehat{K} = 2$	100,0	100,0	100,0
	$\widehat{K} = 4$	100,0	100,0	100,0
	$\widehat{K} = 6$	98,8	99,2	97,1
$Q = 100$	$\widehat{K} = 2$	100,0	98,8	97,5
	$\widehat{K} = 4$	100,0	94,4	92,5
	$\widehat{K} = 6$	95,0	92,1	87,9
$Q = 498$	$\widehat{K} = 2$	100,0	93,8	86,3
	$\widehat{K} = 4$	85,6	76,9	65,0
	$\widehat{K} = 6$	63,8	55,0	52,1

- Giving time to exactly solve the ℓ_0-SU is quite interesting !

Table of contents

Conclusion - Main perspectives

To work with (semi)real datasets: Collaboration with Lucas Drumetz (IMT Atlantique)

- **Unsupervised** SU;
- The dictionary, composed of $Q = 15$ spectra, is learned from a data set;
- **Teaser**
  - Branch-and-bound algorithm is **very promising** in regards to **computation time**;
  - We are in a position to show **the interest of exact optimization**;
  - We have some problems with the **interpretability** of the data in the case of **high correlated** dictionary.

To publish the paper: Ongoing!

To implement other OR stuffs in the current branch-and-bound framework;

To tackle other specific problems: e.g. structured sparsity, . . .

Thank you for your attention

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